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Discrete and continuous graded contractions of representations of Lie algebras

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Abstract. Simultaneous grading of Lie algebras and their representation spaces is used to develop a new theory of grading preserving contractions of representations of all Lie algebras admitting the chosen grading. The theory is completely different from the traditional ways of contracting representations. The graded contractions fall naturally into two classes: discrete and continuous ones related respectively to 2-cocycles and coboundaries of the grading group.

1. Introduction

Contractions of Lie algebras arise as a natural way of passing from one group of symmetries to another similar, but otherwise not directly related group of symmetries. In the traditional approach a contraction of a Lie algebra is the continuous limit of a parametrized family of isomorphic Lie algebras. There is a considerable body of literature on this subject (for excellent expositions see [1-3]) from which it is apparent that the main obstacles to a satisfactory theory are twofold. First, in its full generality the study of all continuous deformations of the structure constants of a Lie algebra [4-6] offers such a bewildering array of possibilities that one cannot hope to obtain precise and practical information except in the simplest cases. Second, to be useful, contractions of Lie algebras need to be accompanied by contractions of their representations, a task that has not proved to be at all easy [7-14].

In [15] it was shown that by working within the context of graded Lie algebras a very straightforward approach to contractions can be made, which is at once general enough to contain all the well known contractions as well as infinitely many others and, at the same time, sufficiently constrained to allow a complete classification once a grading group has been specified.

In this paper we show two things:

(1) This graded contraction process is functorial and depends on the grading group G rather than on the details of structure (for instance, finite or infinite dimensional) of the Lie algebra itself. The contractions are classified by certain weak cohomology classes of G with coefficients in the ground field K and the contractions that fall within the traditional limit process correspond to those cohomology classes that lie in the closure of the coboundary classes.

(2) By considering along with the graded Lie algebras their compatibly graded representations, we obtain a theory of contractions of representations that contains

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the Lie algebra contractions as the special case for the adjoint representation. Our theory is completely different from the traditional approach to contractions of representations [7–14].

The idea of making the study of gradings the backbone of a general approach to Lie theory was apparently put forward only recently [16]. This article can be considered as another example of how fruitful such an approach can be (see also [15–20]).

In section 2, to exemplify what is to follow, we consider the case of $\mathbb{Z}_2$-graded contractions of representations of general Lie algebras over $\mathbb{C}$. Here all the computations are transparent and easy to perform, but nevertheless, the outcome of the representation contractions summarized in table 1 is new.

In section 3 we identify the parameters of the Lie algebra contractions as the values of 2-cocycles of the grading group in a central extension of the group to a semigroup.

In section 4 the category S-Lie(K) of Lie algebras over K with a grading given by an Abelian semigroup S is introduced as well as the contraction functor Γ_ε acting in S-Lie(K). Discrete and continuous contractions are then related respectively to 2-cocycles and coboundaries of S.

In section 5 we study the contractions of representations which are graded in a way compatible with the grading of Lie algebras.

In section 6 we consider the cyclic groups $\mathbb{Z}_n$ as the grading groups. All the graded contractions of representations of $\mathbb{Z}_3$-graded Lie algebras are classified.

In section 7 we show how to compare the contraction of the tensor product of two representations with the tensor product of their contractions. In general, these are quite different and the process shows how we can infer information on the one from the other.

2. Contractions of $\mathbb{Z}_2$-graded representations of Lie algebras

As an illustration and motivation of the general procedure described subsequently, we devote this section to the simplest case, where all the constructions can be made in a straightforward and explicit way. First let us recall the $\mathbb{Z}_2$-graded contractions following [15]. We consider any Lie algebra L of finite or infinite dimension which is graded by the cyclic group $\mathbb{Z}_2$ of two elements. Thus we have

$$L = L_0 \oplus L_1 \tag{2.1}$$

$$0 \neq [L_j, L_k] \subseteq L_{j+k} \qquad j, k, j+k \pmod 2. \tag{2.2}$$

Note that we have chosen to consider the *generic case* where no commutator vanishes identically in (2.2). More precisely, we write $0 \neq [L_j, L_k]$ if there are some elements $x \in L_j$ and $y \in L_k$ such that $0 \neq [x, y]$.

The most general contraction

$$L \to L^\varepsilon \tag{2.3}$$

of L that preserves the $\mathbb{Z}_2$-grading is described in terms of the commutators (2.2) of L (before the contraction) and the matrix $\varepsilon = (\varepsilon_{jk}) \in \mathbb{C}^{2\times 2}$ of contraction parameters introduced as follows:

$$[x, y]_\varepsilon := \varepsilon_{j,k}[x, y] \qquad \text{for } x \in L_j, y \in L_k. \tag{2.4}$$

As a shorthand we will usually write expressions like (2.4) in the form

$$[L_j, L_k]_\varepsilon := \varepsilon_{jk}[L_j, L_k] \subseteq \varepsilon_{jk} L_{j+k}. \tag{2.4'}$$

Here the subscript ε denotes the contracted commutator.

The Lie algebra L and its contraction L^ε are isomorphic as linear spaces, only the commutation relations in L^ε are modified by (2.4). In order that the result of the contraction L^ε be a Lie algebra, the contraction parameters must satisfy antisymmetry and the Jacobi identity. This amounts in case (2.2) to the requirements

$$\varepsilon_{jk} = \varepsilon_{kj} \qquad \varepsilon_{00}\varepsilon_{01} = \varepsilon_{01}^2 \qquad \varepsilon_{00}\varepsilon_{11} = \varepsilon_{01}\varepsilon_{11}. \tag{2.5}$$

The latter two equations can be expressed in a more compact form

$$\varepsilon_{jk}\varepsilon_{j+k,m} = \varepsilon_{km}\varepsilon_{j,k+m}. \tag{2.6}$$

If we do not assume the generic case of (2.2) then one or more of the equations (2.5) may be unnecessary. For further discussion of non-generic cases see [15].

The equations (2.5) are solved trivially either by

$$\varepsilon = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{(no contraction)} \tag{2.7}$$

or by

$$\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(Abelian } L^\varepsilon\text{)}. \tag{2.8}$$

The non-trivial contractions are given by the remaining solutions of (2.5). These are

$$\varepsilon = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.9}$$

The first two solutions (2.9) of (2.6) can be obtained by a continuous change of ε starting from (2.7) without ever violating (2.5): they are *continuous contractions* of L. The last one, $\varepsilon = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$, is a *discrete contraction* of L.

Suppose that the action of a $\mathbb{Z}_2$-graded L splits an L-module V into the direct sum

$$V = V_0 \oplus V_1 \tag{2.10}$$

where the subspaces V_j, $j = 0, 1$, are defined by the grading property

$$0 \neq L_j V_m \subseteq V_{j+m} \qquad j, m, j+m \pmod 2. \tag{2.11}$$

As in (2.2) we assume the generic situation.

It is instructive to write V and LV as follows

$$V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \qquad LV = \begin{pmatrix} L_0 & L_1 \\ L_1 & L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} L_0 V_0 + L_1 V_1 \\ L_1 V_0 + L_0 V_1 \end{pmatrix}$$

thus making the graded structures explicit.

We would like V to become a module for L^ε. This imposes a modification on the action of L on V. In the same way as we did in (2.4) for the adjoint representation, we introduce new contraction parameters ψ_{jm}. We denote the contracted action of L on V by $L \overset{\psi}{\cdot} V$. Then we have

$$(L_j V_m)^\psi = L_j \overset{\psi}{\cdot} V_m \subseteq \psi_{jm} L_j V_m \tag{2.12}$$

or, equivalently we write

$$L \overset{\psi}{\cdot} V = \begin{pmatrix} L_0 & L_1 \\ L_1 & L_0 \end{pmatrix} \overset{\psi}{\cdot} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi_{00}L_0 & \psi_{11}L_1 \\ \psi_{10}L_1 & \psi_{01}L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} \psi_{00}L_0V_0 + \psi_{11}L_1V_1 \\ \psi_{10}L_1V_0 + \psi_{01}L_0V_1 \end{pmatrix}. \tag{2.13}$$

In particular, for the adjoint action of L^ε on itself, we have $\psi = \varepsilon$ and (2.13) becomes (3.10) of [15]. Note that in general $\psi_{jm} \neq \psi_{mj}$.

The representation defining relations

$$[L_j, L_k]|m\rangle = L_jL_k|m\rangle - L_kL_j|m\rangle \subseteq L_{j+k}|m\rangle \qquad |m\rangle \in V_m,\ m = 0, 1 \tag{2.14}$$

before a contraction $L \to L^\varepsilon$, and

$$[L_j, L_k]_\varepsilon \overset{\psi}{\cdot} |m\rangle = \psi_{km}\psi_{j,k+m}L_jL_k|m\rangle - \psi_{jm}\psi_{k,j+m}L_kL_j|m\rangle$$

$$\subseteq \varepsilon_{jk}\psi_{j+k,m}L_{j+k}|m\rangle \tag{2.15}$$

after the contraction, must hold simultaneously. Thus we arrive at the equations for ψ defining the action of L^ε in V:

$$\varepsilon_{jk}\psi_{j+k,m} = \psi_{km}\psi_{j,k+m} = \psi_{jm}\psi_{k,j+m} \tag{2.16}$$

where the subscripts are read modulo 2 in the $\mathbb{Z}_2$ case considered here. In particular, for the adjoint representation one has $\psi = \varepsilon$, and equations (2.16) coincide with (2.6).

The action of L^ε on V is thus determined by a pair of matrices ε and ψ in $\mathbb{C}^{2\times 2}$.

There are two obvious solutions of (2.16): the trivial one $\psi = (\psi_{jm}) = (0)$ which we disregard, and $\psi = \varepsilon$, which is exemplified by the adjoint representation. In the latter case we find for ψ the solutions given in (2.9). A complete list of non-trivial solutions ψ for each ε is found in table 1.

Note that in the case of the trivial contraction (2.7) of L the system (2.16) of equations for ψ has only trivial solutions:

$$\psi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 1. Consider an example of the simple Lie algebra $sl(3, \mathbb{C})$ and its natural representation (1 0) of dimension 3. We denote by α and β the simple roots. The standard basis of the algebra is given by the generators of the root spaces e_α, f_α, e_β, f_β and their commutators

$$h_\alpha = [e_\alpha, f_\alpha] \qquad h_\beta = [e_\beta, f_\beta]$$
$$e_{\alpha+\beta} = [e_\alpha, e_\beta] \qquad f_{\alpha+\beta} = [f_\alpha, f_\beta]. \tag{2.17}$$

Consider the following $\mathbb{Z}_2$-grading of $sl(3, \mathbb{C})$

$$L_0 = \mathbb{C}h_\alpha + \mathbb{C}h_\beta + \mathbb{C}e_\alpha + \mathbb{C}f_\alpha$$
$$L_1 = \mathbb{C}e_\beta + \mathbb{C}f_\beta + \mathbb{C}e_{\alpha+\beta} + \mathbb{C}f_{\alpha+\beta} \tag{2.18}$$

Table 1. Non-trivial $\mathbb{Z}_2$-graded contractions of representations. The results are presented in the format of equation (2.13).

ε	No.	$L \overset{\psi}{\cdot} V$	
$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$	I.1	$\begin{pmatrix}L_0 & 0\\ L_1 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0\\ L_1V_0+L_0V_1\end{pmatrix}$	$\varepsilon=\psi$
	I.2	$\begin{pmatrix}L_0 & L_1\\ 0 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0+L_1V_1\\ L_0V_1\end{pmatrix}$	
	I.3	$\begin{pmatrix}L_0 & 0\\ 0 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0\\ L_0V_1\end{pmatrix}$	
	I.4	$\begin{pmatrix}L_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0\\ 0\end{pmatrix}$	
	I.5	$\begin{pmatrix}0 & 0\\ 0 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}0\\ L_0V_1\end{pmatrix}$	
$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$	II.1	$\begin{pmatrix}0 & L_1\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_1V_1\\ 0\end{pmatrix}$	$\varepsilon=\psi$
	II.2	$\begin{pmatrix}0 & 0\\ L_1 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}0\\ L_1V_0\end{pmatrix}$	
$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	III.1	$\begin{pmatrix}L_0 & 0\\ 0 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0\\ L_0V_1\end{pmatrix}$	
	III.2	$\begin{pmatrix}L_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_0V_0\\ 0\end{pmatrix}$	$\varepsilon=\psi$
	III.3	$\begin{pmatrix}0 & 0\\ 0 & L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}0\\ L_0V_1\end{pmatrix}$	
	III.4	$\begin{pmatrix}0 & L_1\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}L_1V_1\\ 0\end{pmatrix}$	
	III.5	$\begin{pmatrix}0 & 0\\ L_1 & 0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\end{pmatrix}=\begin{pmatrix}0\\ L_1V_0\end{pmatrix}$	

where $\mathbb{C}$ stands for an arbitrary complex coefficient. Then one can verify directly that

$$\begin{aligned}&[L_0, L_0]=\mathbb{C}h_\alpha+\mathbb{C}e_\alpha+\mathbb{C}f_\alpha\subset L_0\\ &[L_0, L_1]=L_1\\ &[L_1, L_1]=L_0.\end{aligned} \tag{2.19}$$

Consequently the grading is generic.

The representation space V,

$$V=\mathbb{C}|1,0\rangle+\mathbb{C}|-1,1\rangle+\mathbb{C}|0,-1\rangle \tag{2.20}$$

is spanned by the weight vectors $|1,0\rangle$, $|-1,1\rangle$, $|0,-1\rangle$. The graded action of $sl(3,\mathbb{C})$ splits V into two subspaces V_0 and V_1 defined by

$$L_0V_0=V_0 \qquad L_0V_1=V_1 \qquad L_1V_0=V_1 \qquad L_1V_1=V_0 \tag{2.21}$$

where

$$V_0=\mathbb{C}|1,0\rangle+\mathbb{C}|-1,1\rangle \qquad V_1=\mathbb{C}|0,-1\rangle. \tag{2.22}$$

In this very simple example it is equally easy to write the relations (2.21) in a matrix form. One has

$$
\begin{aligned}
L_0 V_0 &\Rightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{pmatrix}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \\ 0 \end{pmatrix} \\
L_0 V_1 &\Rightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(a+d)z \end{pmatrix} \\
L_1 V_0 &\Rightarrow \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ g & h & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ gx+hy \end{pmatrix} \\
L_1 V_1 &\Rightarrow \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ g & h & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} ez \\ fz \\ 0 \end{pmatrix} \\
& a, b, c, d, x, y, z \in \mathbb{C}.
\end{aligned}
\tag{2.23}
$$

Let us now consider the contractions of L with $\varepsilon = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The entries I.1–I.5 of table 1 are the contractions of the representations in this case. The matrix representation of the operations (2.23) can be substituted into the column $L \overset{\psi}{\cdot} V$ of table 1 in order to have the contracted algebra and its action on V written in a matrix form. Thus for example, for I.1 we get

$$
L \overset{\psi}{\cdot} V = \begin{pmatrix} L_0 & 0 \\ L_1 & L_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & -a-d & 0 & 0 & 0 \\ 0 & 0 & e & a & b & 0 \\ 0 & 0 & f & c & d & 0 \\ g & h & 0 & 0 & 0 & -a-d \end{pmatrix}\begin{pmatrix} x \\ y \\ 0 \\ 0 \\ 0 \\ z \end{pmatrix}.
$$

Without loss of generality this can be rewritten as

$$
L \overset{\psi}{\cdot} V = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ g & h & -a-d \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \\ gx+hy-(a+d)z \end{pmatrix}. \tag{2.24}
$$

Similarly for I.2 of table 1, we would find

$$
L \overset{\psi}{\cdot} V = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & -a-d \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+ez \\ cx+dy+fz \\ -(a+d)z \end{pmatrix}. \tag{2.25}
$$

For representations of higher dimensions it is not practical to write the matrices explicitly. It suffices to describe the subspaces V_0 and V_1 and to use table 1.

This example requires a comment. Here the grading of sl(3, ℂ) and V is the result of the action of a cyclic subgroup of SU(3) ⊂ SL(3, ℂ), which is generated by an element g of the SU(3)-conjugacy class [1 0 1] (for properties and further details of these notations see [21] and [22]). The element g acts on irreducible representations of sl(3, ℂ) with eigenvalues ± 1 (representations of congruence class 0) and $\pm\sqrt{-1}$ (representation of congruence classes 1 and 2). Having considered in our example a single irreducible representation, we could simplify the example by using $\mathrm{sl}(3, \mathbb{C}) = L_0 \oplus L_1$ and $V = V_0 \oplus V_1$ for what should have been $\mathrm{sl}(3, \mathbb{C}) = L_0 \oplus L_2$ and $V = V_1 \oplus V_3$ (subscripts mod 4). In general, when tensor products of representations are considered (section 7), such a simplification would lead to inconsistencies in the grading.

3. Semigroups and 2-cocycles

Our theory of contractions is based on a generalization of the theory of 2-cocycles as they appear in group cohomology and particularly in the theory of central extensions of groups.

Briefly, if G is a group, then a central extension of G by an Abelian group K is a group $\hat{G}$ whose centre contains K and for which there is a surjective homomorphism $\pi : \hat{G} \to G$ whose kernel is K. Given such a central extension let us take any *section* $\alpha : G \to \hat{G}$, that is a map satisfying $\pi \cdot \alpha = \mathrm{id}_G$. Given any $g, h \in G$,

$$\alpha(g)\alpha(h)\alpha(gh)^{-1} \overset{\pi}{\mapsto} 1.$$

So we introduce

$$\varepsilon_{g,h} := \alpha(g)\alpha(h)\alpha(gh)^{-1} \in K \tag{3.1}$$

defining a map

$$\varepsilon : G \times G \to K \qquad (g, h) \mapsto \varepsilon_{g,h} \tag{3.2}$$

for which we have

$$\varepsilon_{g,h}\alpha(gh) = \alpha(g)\alpha(h). \tag{3.3}$$

From $\alpha((gh)k) = \alpha(g(hk))$ we obtain at once that

$$\varepsilon_{g,h}\varepsilon_{gh,k} = \varepsilon_{g,hk}\varepsilon_{h,k}. \tag{3.4}$$

These equations differ from (2.6) in that there we are treating the Abelian grading group multiplicatively as opposed to the additive interpretation there.

A mapping (3.2) satisfying (3.4) is a *2-cocycle* on G with values in K. Conversely, given such a map we can construct a central extension $\hat{G}^{\varepsilon}$ of G by K setting

$$\hat{G}^{\varepsilon} = G \times K \qquad \text{(as a set)} \tag{3.5}$$

and defining multiplication by

$$(g, a)(h, b) = (gh, \varepsilon_{g,h}ab) \tag{3.6}$$

and the projection

$$\pi_{\varepsilon} : \hat{G} \to G \qquad (g, a) \mapsto g. \tag{3.7}$$

The multiplication is associative by (3.4). It is easy to see that $\varepsilon_{1,g} = \varepsilon_{g,1}$ is an element e of K independent of g and $(1, e^{-1})$ is the identity of $\hat{G}^{\varepsilon}$ and $(g^{-1}, e^{-1}\varepsilon_{g^{-1},g}^{-1}a^{-1})$ is the inverse of (g, a).

Now let G be any Abelian group and let K be a field treated as a monoid under multiplication. The non-zero elements of K are denoted by $K^\times$. (Much of what we are going to say would work for any group G, not necessarily Abelian, and any commutative monoid.) A *weak 2-cocycle* on G with values in the monoid K is a map (3.2) satisfying (3.4). We denote the set of weak 2-cocycles by $C^2(G, K)$. Given $\varepsilon \in C^2(G, K)$ we construct $\hat{G}$ and a multiplication on it by (3.5) and (3.6). This makes $\hat{G}^\varepsilon$ into a semigroup. Notice that

$$\hat{G}^\varepsilon = \hat{G}_0^\varepsilon \cup \hat{G}_\times^\varepsilon \tag{3.8}$$

where

$$\hat{G}_0^\varepsilon = G \times \{0\} \qquad \hat{G}_\times^\varepsilon = G \times K^\times$$

and $\hat{G}_0^\varepsilon$ is a subsemigroup of $\hat{G}^\varepsilon$ isomorphic to $\hat{G}$. If ε only takes values in $K^\times$ then ε is called ***regular*** and it is clearly a 2-cocycle on G with coefficients in $K^\times$ and $\hat{G}_\times^\varepsilon$ is the corresponding extension of G by $K^\times$. However, from the point of view of our theory of contractions the interest lies in the cocycles that are not regular.

If ε, $\varepsilon' \in C^2(G, K)$ then their product

$$\varepsilon\varepsilon' : G \times G \to K \qquad (\varepsilon\varepsilon')_{g,h} \mapsto \varepsilon_{g,h}\varepsilon'_{g,h} \tag{3.9}$$

is also in $C^2(G, K)$ and thus $C^2(G, K)$ is a commutative semigroup.

Suppose that $\varepsilon \in C^2(G, K)$ and

$$\hat{G}^\varepsilon \xrightarrow{\pi_\varepsilon} G$$

is the corresponding extension. Further suppose that

$$\alpha : G \to \hat{G}^\varepsilon \qquad g \mapsto (g, a_g) \qquad a_g \in K^\times$$

is some section. Then analogously to (3.3), computing $\alpha(gh)$ and $\alpha(g)\alpha(h)$,

$$\varepsilon'_{g,h} a_{gh} = \varepsilon_{g,h} a_g a_h \tag{3.10}$$

so we have a new weak 2-cocycle

$$\varepsilon'_{g,h} := \varepsilon_{g,h} \frac{a_g a_h}{a_{gh}}. \tag{3.11}$$

A 1-cochain on G is a map

$$a : G \to K. \tag{3.12}$$

It is *regular* if $a(G) \subset K^\times$. If a is regular then we can define $\mathrm{d}a \in C^2(G, K)$ by

$$\mathrm{d}a_{g,h} = a_g a_h a_{gh}^{-1}. \tag{3.13}$$

The set of 2-cocycles of this type form a subgroup $B^2(G, K)$ of $C^2(G, K)$.

If $\varepsilon \in C^2(G, K)$ and a is a regular 1-cochain then, with $\varepsilon' := \varepsilon\, \mathrm{d}a$, we have an isomorphism

$$\varphi : \hat{G}^{\varepsilon'} \to \hat{G}^\varepsilon \qquad \text{over } G \tag{3.14}$$

that is

$$\begin{array}{ccc} \hat{G}^{\varepsilon'} & \xrightarrow{\varphi} & \hat{G}^{\varepsilon} \\ {\scriptstyle \pi'_\varepsilon}\downarrow & & \downarrow{\scriptstyle \pi_\varepsilon} \\ G & = & G \end{array}$$

given by

$$(g, c) \xrightarrow{\varphi} (g, a_g c).$$

In fact,

$$\varphi((g, c)(g', c')) = \varphi(gg', \varepsilon'_{g,g'}cc') = \varphi(gg', a_g a_{g'} \varepsilon_{g,g'} a_{gg'}^{-1} cc')$$
$$= \varphi(gg', a_g a_{g'} \varepsilon_{g,g'} cc') = \varphi((g, c))\varphi((g', c')). \tag{3.15}$$

The import of (3.11) and (3.14) is that multiplying a cocycle ε by an element of $B^2(G, K)$ is equivalent to choosing a different section of $\hat{G}^\varepsilon$. In the examples we make a free use of this in classifying cocycles.

More formally, we define an equivalence relation $\sim$ on C^2 by

$$\varepsilon' \sim \varepsilon \qquad \text{if and only if } \varepsilon' = \varepsilon \, \mathrm{d}a \tag{3.16}$$

for some regular 1-cochain a. It is easy to see that

$$\varepsilon_1 \sim \varepsilon_2 \qquad \varepsilon'_1 \sim \varepsilon'_2 \Rightarrow \varepsilon_1 \varepsilon'_1 \sim \varepsilon_2 \varepsilon'_2. \tag{3.17}$$

Thus we may form the quotient semigroup

$$H^2(G, K) := (C^2(G, K)/\sim). \tag{3.18}$$

We are primarily interested in $H^2(G, K)$.

In spite of appearances the theory of semigroup extensions $\hat{G}^\varepsilon$ is considerably more complex than the corresponding theory of group extensions. For example, $\hat{G}^\varepsilon$ need not have an identity element and the subgroup $\hat{G}_0^\varepsilon$, which is rather like a 'sink', has no counterpart in group theory.

Lemma. $H := \hat{G}_0^\varepsilon$ is characterized in $\hat{G}^\varepsilon$ by the following properties:

(i) $\pi_\varepsilon|_H$ is an isomorphism onto G;

(ii) $H\hat{G}^\varepsilon = H$.

Proof. Clearly $\hat{G}_0^\varepsilon$ satisfies (i) and (ii). Conversely let H satisfy (i) and (ii). Then from (i) each $\hat{h} \in H$ is uniquely expressible as $(h, a(h))$ where $h = \pi_\varepsilon(\hat{h})$ and $a(h)$ is some element of K. But for $(g, c) \in \hat{G}^\varepsilon$,

$$(h, a(h))(g, c) = (hg, \varepsilon_{h,g} a(h)c) = (hg, a(hg))$$

by (ii). Thus $a(hg) = \varepsilon_{h,g} a(h)c$ independently of $c \in K$. Thus $a(hg) = 0$. Setting $g = 1$ gives $a(h) = 0$ so $(h, a(h)) \in \hat{G}_0^\varepsilon$. □

In the special case that $K = \mathbb{R}$ or $\mathbb{C}$ we may topologize $C^2(G, K)$ using the metric

$$\|\varepsilon - \varepsilon'\| = \sup_{(g,h) \in G \times G} \|\varepsilon_{g,h} - \varepsilon'_{g,h}\|.$$

Evidently C^2 is a closed set.

We say that

$$\varepsilon \in C^2 \text{ is a } \textit{limit cocycle} \text{ or } \textit{Wigner–Inönü cycle} \tag{3.19}$$

if $\varepsilon \in \overline{B^2}$ (the closure of B^2 in C^2); ε is a non-trivial limit cycle if $\varepsilon \in (\overline{B^2}\backslash B^2)$.

Proposition. Suppose that G is a finite group of order N and

$$\varepsilon = \lim_{n\to\infty} \varepsilon^{(n)} \qquad \varepsilon^{(n)} \in B^2$$

is a non-trivial limit cocycle. Then ε is not regular.

Proof. We write $\varepsilon^{(n)}_{g,h} = a^{(n)}_g a^{(n)}_h / a^{(n)}_{gh}$, $g, h \in G$. If $\lim_{n\to\infty} a^{(n)}_g = \bar{a}_g \neq 0$ exists for each $g \in G$ then evidently $\varepsilon_{g,h} = \bar{a}_g \bar{a}_h / \bar{a}_{gh}$ and $\varepsilon \in B^2$.

Suppose that ε is regular. With $h = 1$ we obtain $0 \neq \varepsilon_{g,1} = \lim_{n\to\infty} a^{(n)}_1$. Let $g \in G$ have order k. Then for some $\alpha \neq 0$

$$\varepsilon^{(n)}_{g,g} \varepsilon^{(n)}_{g,g^2} \ldots \varepsilon^{(n)}_{g,g^{k-1}} \xrightarrow[n\to\infty]{} \alpha$$

and hence

$$\frac{a^{(n)}_g a^{(n)}_g}{a^{(n)}_{g^2}} \cdot \frac{a^{(n)}_g a^{(n)}_{g^2}}{a^{(n)}_{g^3}} \cdots \frac{a^{(n)}_g a^{(n)}_{g^{k-1}}}{a^{(n)}_1} = \frac{(a^{(n)}_g)^k}{a^{(n)}_1} \xrightarrow[n\to\infty]{} \alpha \neq 0$$

and so $\beta_g := \lim_{n\to\infty} (a^{(n)}_g)^k \neq 0$ exists.

Let γ_g be any fixed kth-root of β_g for each g, and let U_N be the group of Nth roots of 1 in K. Then there is a map

$$f_g : \mathbb{Z}_+ \to U_N$$

so that

$$\{f_g(n) a^{(n)}_g\}_n \to \gamma_g.$$

Since G and U_N are finite there is an infinite subsequence $\{f_g(n) a^{(n)}_g\}_{n\in S}$ on which $f_g(n) = \bar{f}_g$ is independent of n. Thus $\{\bar{f}_g a^{(n)}_g\}_{n\in S}$ converges to γ_g.

Now

$$\left\{\frac{\bar{f}_g a^{(n)}_g \bar{f}_h a^{(n)}_h}{\bar{f}_{gh} a^{(n)}_{gh}}\right\}_{n\in S} \to \frac{\bar{f}_g \bar{f}_h}{\bar{f}_{gh}} \varepsilon_{g,h} = \frac{\gamma_g \gamma_h}{\gamma_{gh}}.$$

Thus $\varepsilon_{g,h} = \bar{a}_g \bar{a}_h / \bar{a}_{gh}$ where $\bar{a}_g := f_g^{-1} \gamma_g$, and therefore ε is trivial. □

Up until now we have not required that our extension semigroups $\hat{G}_\varepsilon$ be commutative. In the sequel we will wish this to be true. This amounts to the requirement

$$\varepsilon_{g,h} = \varepsilon_{h,g} \qquad \text{for all } g, h \in G. \tag{3.20}$$

The set of 2-cocycles of this form is denoted by $C^2_+(G, K)$. We clearly have $B^2 \subset C^2_+(G, K)$ and hence a subsemigroup

$$H^2_+(G, K) = (C^2_+(G, K)/\sim) \subset H^2(G, K). \tag{3.21}$$

4. Contractions of Lie algebras

Let S be a commutative semigroup. We let S-Lie(K) denote the category of all Lie algebras over K that are graded by S, namely

$$L \in S\text{-Lie}(K) \Leftrightarrow L = \bigoplus_{s \in S} L_s$$

for some subspaces L_s of L and

$$[L_s, L_t] \in L_{st}$$

with morphisms being Lie algebra homomorphisms that preserve the grading (i.e. for L, $L' \in S$-Lie(K), $\varphi : L \to L'$ should satisfy $\varphi L_s \subset L'_s$). It was shown in [16] that for a simple Lie algebra the grading semigroup is a group.

Let G be an Abelian group and let $\varepsilon \in C^2_+(G, K)$. We define a functor Γ_ε on G-Lie(K) as follows: for $L \in G$-Lie(K), $L^\varepsilon := \Gamma_\varepsilon(L)$ is the Lie algebra with

(i) vector space structure equal to L

(ii) multiplication defined by L

$$[x, y]_\varepsilon = \varepsilon_{g,h}[x, y] \qquad g, h \in G, x \in L_g, y \in L_h. \tag{4.1}$$

The skew symmetry and the Jacobi identity are immediate consequences of $\varepsilon \in C^2_+(G, K)$. It is clear that if L, $L' \in G$-Lie(K) and $\varphi : L \to L'$ is a homomorphism then there is a canonical homomorphism

$$\Gamma_\varepsilon(\varphi) : L^\varepsilon \to L'^\varepsilon. \tag{4.2}$$

Remark. The requirement that $\varepsilon \in C^2_+(G, K)$ is, from a generic point of view, a necessary condition for (4.1) to work.

Indeed, let $\varepsilon : G \times G \to K$ be an arbitrary map and use (4.1) to construct an algebra L^ε from L. Suppose that L^ε is a Lie algebra. Then for $x \in L_g$, $y \in L_h$, $z \in L_k$ we have

$$\varepsilon_{g,h}[x, y] = [x, y]_\varepsilon = -[y, x]_\varepsilon = -\varepsilon_{h,g}[y, x] \tag{4.3}$$

and

$$\varepsilon_{g,hk}\varepsilon_{h,k}[x, [y, z]] + \varepsilon_{h,kg}\varepsilon_{k,g}[y, [z, x]] + \varepsilon_{k,gh}\varepsilon_{g,h}[z, [x, y]] = 0. \tag{4.4}$$

If we assume that $[x, y] \neq 0$ and the pair $[x, [y, z]]$ and $[y, [z, x]]$ are linearly independent then (3.4) and (3.20) follow and $\varepsilon \in C^2_+(G, K)$.

If ε, $\varepsilon' \in C^2_+(G, K)$ and $\varepsilon' \sim \varepsilon$ then the functors Γ_ε and Γ'_ε are naturally equivalent, i.e. for each $L \in G$-Lie(K) there is an isomorphism

$$\mu_L : \Gamma_{\varepsilon'}(L) \to \Gamma_\varepsilon(L) \tag{4.5}$$

so that whenever $\varphi : L \to L'$ is an isomorphism we have

$$\begin{array}{ccc} \Gamma_{\varepsilon'}(L) & \xrightarrow{\mu_L} & \Gamma_\varepsilon(L) \\ {\scriptstyle \Gamma_{\varepsilon'}(\varphi)}\downarrow & & \downarrow{\scriptstyle \Gamma_\varepsilon(\varphi)} \\ \Gamma_{\varepsilon'}(L') & \xrightarrow{\mu_{L'}} & \Gamma_\varepsilon(L') \end{array} \tag{4.6}$$

For by assumption $\varepsilon' = \varepsilon\,\mathrm{d}a$ for some regular 1-cocycle a. The map μ_L defined by $x \mapsto a_g x$ for all $x \in L_g$, $g \in G$, works:

$$\begin{aligned} \mu_L([x, y]_{\varepsilon'}) &= \mu_L(\varepsilon'_{g,h}[x, y]) = a_{gh}\varepsilon'_{g,h}[x, y] \\ &= a_g a_h \varepsilon_{g,h}[x, y] = [\mu_L(x), \mu_L(y)]_\varepsilon. \end{aligned}$$

It is interesting to see what the functor Γ_ε means when ε is a limit cocycle. Then

$$\varepsilon = \lim_{n\to\infty} \varepsilon^{(n)} \qquad \varepsilon^{(n)}_{g,h} = a^{(n)}_g a^{(n)}_h / a^{(n)}_{gh}$$

with regular 1-cochain $a^{(n)}_g$. The map $x \mapsto a^{(n)}_g x$, $x \in L_g$, $g \in G$, is an isomorphism between $L^{\varepsilon^{(n)}}$ and L. But in the limit we have

$$[x, y]_\varepsilon = \lim_{n\to\infty} \frac{a^{(n)}_g a^{(n)}_h}{a^{(n)}_{gh}} [x, y]$$

and in general $L^\varepsilon \not\cong L$. For instance, if G is finite and ε is non-trivial (3.19), then by the proposition $\varepsilon_{g,h}$ will vanish for certain g, h whereas $\varepsilon^{(n)}_{g,h} \neq 0$.

The process of contraction by limits of boundary cocycles is the standard definition found in literature [1-3] called the Wigner–Inönü contraction. Thus these standard contractions are those coming from the elements of $(\overline{B^2/\sim})$ in $H^2_+(G, C)$.

Definition. A contraction Γ_ε is of *continuous type* if $\varepsilon \in (\overline{B^2/\sim})$. It is of *discrete type* otherwise.

Thus we can observe that the Wigner–Inönü contractions arise as limits of the coboundaries of the grading group which in almost all cases in the literature has been the cyclic group of two elements. This is only a fraction of the possibilities if one admits ε to be any 2-cocycle of the grading group.

It is interesting to observe that given L and its contraction L^ε there is a canonically associated Lie algebra $\hat{L}^\varepsilon$ that has both as homomorphic images. Namely, we define a vector space

$$\hat{L}^\varepsilon = \bigoplus_{(g,a)\in\hat{G}^\varepsilon} L_{(g,a)} \tag{4.7}$$

where $L_{(g,a)}$ is a vector space isomorphic to $L_g \subset L$. We denote the element of $L_{(g,a)}$ associated by this isomorphism to $x \in L_g$ by (x, a). We define multiplication in $\hat{L}^\varepsilon$ by

$$[(x, a), (y, b)] = ([x, y], \varepsilon_{g,h} ab) \tag{4.8}$$

for $x \in L_g$, $y \in L_h$; $a, b \in K$. This indeed defines a Lie algebra structure on $\hat{L}^\varepsilon$ and we have homomorphisms

$$\pi : \hat{L}^\varepsilon \to L \qquad (x, a) \mapsto x \tag{4.9}$$

and

$$\pi_\varepsilon : \hat{L}^\varepsilon \to L^\varepsilon \qquad (x, a) \mapsto ax. \tag{4.10}$$

The map $\Lambda^\varepsilon : L \to \hat{L}^\varepsilon$ is a functor from G-Lie$(K) \to \hat{G}^\varepsilon$-Lie$(K)$.

5. Contractions of modules

We now show how to define a theory of contractions of representations in terms of Lie algebra contractions. Let M be a non-empty set and let V be a vector space over K. We say that V is M-graded if

$$V = \bigoplus_{m\in M} V_m. \tag{5.1}$$

Let G be a group and suppose that G acts on M, i.e. there is a map

$$G \times M \to M \qquad (g, m) \mapsto g \cdot m \tag{5.2}$$

such that

$$g \cdot (h \cdot m) = (gh) \cdot m \qquad g, h \in G, m \in M. \tag{5.3}$$

Suppose that $L \in G\text{-Lie}(K)$. We define the category $M\text{-Mod}(L)$ to be the set of all L-modules V that satisfy

$$\begin{array}{lll} \text{(i)} & V \text{ is graded by } M & \\ \text{(ii)} & L_g \cdot V_m \subset V_{g \cdot m} & g \in G, m \in M \end{array} \tag{5.4}$$

together with the set of all L-module maps that preserve the M-grading.

Let G and M be as above and let $\varepsilon \in C^2_+(G, K)$. We define the set $F(M, \varepsilon)$ of all maps

$$\psi : G \times M \to M \tag{5.5}$$

such that

$$\varepsilon_{g,h}\psi_{gh,m} = \psi_{g,hm}\psi_{h,m} = \psi_{g,m}\psi_{h,gm}. \tag{5.6}$$

This is the multiplicative form of (2.16). A special case of this is $G = M$, $\psi = \varepsilon$ whereupon (5.6) is simply (3.4).

Given $\psi \in F(M, \varepsilon)$ and $V \in M\text{-Mod}(L)$ we define V^ψ to be the L^ε-module with
(i) vector space structure equal to V;
(ii) action of L^ε defined by

$$x \overset{\psi}{\cdot} v = \psi_{g,m} x \cdot v \qquad x \in (L^\varepsilon)_g, v \in V_m. \tag{5.7}$$

We call V^ψ the *ψ-contraction* of V *relative* to ε.

Remarks.

(1) Just as in the remark in section 4, one can see that from a generic point of view a map (5.5) will provide a module structure for L^ε via (5.7) if and only if (5.6) holds, i.e. $\psi \in F(M, \varepsilon)$.

(2) In sections 2 and 6 we have determined all the possible ψ-contractions as ε runs over all elements of $H^2_+(G, \mathbb{C})$ for $G = \mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively.

(3) When $M = G$, $\psi = \varepsilon$ and $V = L$ we obtain the adjoint representation of L^ε as a contraction of the adjoint representation of L.

(4) The association $V \to V^\psi$ is in fact a functor

$$\Gamma^\psi_\varepsilon : M\text{-Mod}(L) \to M\text{-Mod}(L^\varepsilon).$$

Let $\psi \in F(M, \varepsilon)$. We define

$$\hat{M}^\psi := M \times K \tag{5.8}$$

and define an action of $\hat{G}^\varepsilon$ on $\hat{M}^\psi$ by

$$(g, a) \overset{\psi}{\cdot} (m, c) = (g \cdot m, \psi_{g,m} ac). \tag{5.9}$$

It follows from (5.6) that it is indeed an action.

Observe also that $\hat{M}^{\psi}$ admits a K-action:

$$a(m, c) = (m, ac) \tag{5.10}$$

and that for all $\hat{g} \in \hat{G}^{\varepsilon}$, $\hat{m} \in \hat{M}^{\psi}$, $a \in K$,

$$\hat{g} \overset{\psi}{\cdot} (a\hat{m}) = a(\hat{g} \overset{\psi}{\cdot} \hat{m}). \tag{5.11}$$

Now one may look at the pair (ε, ψ) in terms of the two sections

$$\alpha : G \to \hat{G}^{\varepsilon} \qquad g \mapsto (g, 1) \tag{5.12}$$

$$\beta : M \to \hat{M}^{\psi} \qquad m \mapsto (m, 1) \tag{5.13}$$

for we have

$$\alpha(g) \overset{\psi}{\cdot} \beta(m) = \psi_{g,m}\beta(gm) \tag{5.14}$$

i.e.

$$(g, 1) \overset{\psi}{\cdot} (m, 1) = (g \cdot m, \psi_{g,m}). \tag{5.15}$$

Just as in section 3 we defined equivalent (cohomologous) cocycles by looking at the effect of changing sections, so we obtain the notion of equivalence pairs (ε, ψ) by changing the sections for G and M.

Suppose that

$$a : G \to K^{\times} \tag{5.16}$$

$$b : M \to K^{\times} \tag{5.17}$$

are arbitrary maps and we define new sections

$$\alpha' : G \to \hat{G}^{\varepsilon} \qquad g \mapsto (g, a_g)$$

$$\beta' : M \to \hat{M}^{\psi} \qquad m \mapsto (m, b_m)$$

and define $\psi' : G \times M \to M$ by

$$\alpha'(g) \overset{\psi}{\cdot} \beta'(m) = \psi'_{g,m}\beta'(g \cdot m). \tag{5.18}$$

Thus

$$(g, a_g) \overset{\psi}{\cdot} (m, b_m) = \psi'_{g,m}(g \cdot m, b_{gm}) \tag{5.19}$$

and we have

$$\psi'_{g,m} = \frac{a_g b_m}{b_{gm}} \psi_{g,m}. \tag{5.20}$$

Recall also (3.11). One sees that $\psi' \in F(M, \varepsilon')$ so $\hat{M}^{\psi'}$ is $\hat{G}_{\varepsilon'}$-set. In that case we have the isomorphism $\varphi : \hat{G}_{\varepsilon'} \to \hat{G}_{\varepsilon}$ of (3.14) and also an isomorphism

$$\lambda : \hat{M}^{\psi'} \to \hat{M}^{\psi} \qquad \lambda : (m, c) \mapsto (m, b_m c)$$

and the commutative diagram

$$\begin{array}{ccc} \hat{G}^{\varepsilon'} \times \hat{M}^{\psi'} & \longrightarrow & \hat{M}^{\psi'} \\ \varphi \downarrow \quad \lambda \downarrow & & \downarrow \lambda \\ \hat{G}^{\varepsilon} \times \hat{M}^{\psi} & \longrightarrow & \hat{M}^{\psi} \end{array} \tag{5.21}$$

Given ε, $\varepsilon' \in C^2_+(G, K)$, $\psi \in F(M, \varepsilon)$, $\psi' \in F(M, \varepsilon')$, we write $(\varepsilon, \psi) \sim (\varepsilon', \psi')$ if a and b of (5.16) and (5.17) exist so that $\varepsilon = \varepsilon\, \mathrm{d}a$ and (5.20) holds.

We return to contractions. Suppose that $(\varepsilon', \psi') \sim (\varepsilon, \psi)$. For each Lie algebra $L \in G\text{-Lie}(K)$, the contractions $L^{\varepsilon'}$ and L^{ε} are canonically isomorphic (4.6) and by this isomorphism we may identify them. Given $V \in M\text{-Mod}(L)$, let

$$V^{\psi'} := \Gamma_{\varepsilon'}^{\psi'}(V) \qquad V^{\psi} := \Gamma_{\varepsilon}^{\psi}(V). \tag{5.22}$$

The map

$$\begin{aligned} \nu_V &: V^{\psi'} \to V^{\psi} \\ \nu_V &: v \mapsto b_m v \qquad v \in V_m \end{aligned} \tag{5.23}$$

is an isomorphism of Lie modules ($L^{\varepsilon'}$ and L^{ε} being identified) by (5.20).

To summarize: in determining all contractions (ε, ψ) we are free to renormalize the grading subspaces L_g of L by non-zero constants a_g, $g \in G$, and similarly renormalize the grading subspaces V_m of V by non-zero constants b_m, $m \in M$. Full use of this is made to simplify the results of the examples in sections 2 and 6.

Analogously to (4.7) we can define a lifting functor

$$\Lambda_{\varepsilon}^{\psi} : M\text{-Mod}(L) \to \hat{M}^{\varepsilon}\text{-Mod}(\hat{L}^{\varepsilon}). \tag{5.24}$$

For each $V \in M\text{-Mod}(L)$ define

$$\Lambda_{\varepsilon}^{\psi}(V) = \hat{V}^{\psi} := \bigoplus_{(m,a) \in \hat{M}^{\varepsilon}} V_{(m,a)} \tag{5.25}$$

where $V_{(m,a)}$ is a vector space isomorphic to V_m by some map

$$(v, a) \leftrightarrow v \in V_m.$$

Then $\hat{V}^{\psi}$ is made into an $\hat{L}^{\varepsilon}$-module by

$$(x, a) \overset{\psi}{\cdot} (v, c) = (x \cdot v, \psi_{g,m} ac) \tag{5.26}$$

for $x \in L_g$, $v \in V_m$ and a, $c \in K$. Thus we have maps

$$\begin{aligned} \lambda &: \hat{V}^{\psi} \to V \qquad (v, c) \mapsto v \\ \lambda^{\psi} &: \hat{V}^{\psi} \to V^{\psi} \qquad (v, c) \mapsto cv \end{aligned} \tag{5.27}$$

which are compatible with (4.9) and (4.10). In other words V and V^{ψ} may be seen as homomorphic images of $\hat{V}^{\psi}$.

It is also possible to contract the universal enveloping algebra of a Lie algebra $L \in G\text{-Lie}(K)$. Let $\varepsilon \in H_{+}^{2}(G, K)$. The grading of L gives $U(L)$ a G-grading

$$U = \bigoplus_{g \in G} U_g. \tag{5.28}$$

The contraction $U^{\varepsilon}(L)$ is:

(i) $U^{\varepsilon}(L) = U$ as a vector space;

(ii) multiplication is defined by

$$x \overset{\psi}{\cdot} y = \varepsilon_{g,h} xy \qquad x \in U_g, y \in U_h. \tag{5.29}$$

Clearly $U^{\varepsilon}(L)$ is not in general the universal enveloping algebra $U(L^{\varepsilon})$ since $\varepsilon_{g,h}$ may be zero, but $U(L^{\varepsilon})$ has no zero divisors. Furthermore, $L^{\varepsilon} \subset U^{\varepsilon}(L)$ may not generate $U^{\varepsilon}(L)$ as an algebra. Nevertheless, if $\psi \in F(M, \varepsilon)$ and V^{ψ} is the contraction of the L-module V then V^{ψ} is naturally a $U^{\varepsilon}(L)$ module via

$$u \overset{\psi}{\cdot} v = \psi_{g,m} uv \qquad u \in U_g, v \in V_m. \tag{5.30}$$

As an example of what is involved when inferring information about $U^\varepsilon(L)$ from $U(L)$, let us consider the question of quadratic Casimir operators.

Suppose that

$$C=\sum_{g,i} x_g^{(i)} y_{g^{-1}}^{(i)} \qquad g\in G, x_g^{(i)}\in L_g, y_{g^{-1}}^{(i)}\in L_{g^{-1}} \tag{5.31}$$

where g runs over G, is a Casimir operator of L. We define C^ε to be defined by the same expression as seen in $U^\varepsilon(L)$. Then for $z\in L_h$,

$$\begin{aligned}\left[\sum_{g,i} x_g^{(i)}\cdot y_{g^{-1}}^{(i)}, z\right]_\varepsilon &=\sum\{[x_g^{(i)}, z]_\varepsilon\cdot y_{g^{-1}}^{(i)}+x_g^{(i)}\cdot[y_{g^{-1}}^{(i)}, z]_\varepsilon\}\\ &=\sum\{\varepsilon_{g,h}\varepsilon_{gh,g^{-1}}[x_g^{(i)}, z]y_{g^{-1}}^{(i)}+\varepsilon_{g,g^{-1}h}\varepsilon_{g^{-1},h}x_g^{(i)}\cdot[y_{g^{-1}}^{(i)}, z]\}\\ &=\sum\varepsilon_{g,h}\varepsilon_{gh,g^{-1}}\{[x_g^{(i)}, z]y_{g^{-1}}^{(i)}+x_g^{(i)}\cdot[y_{g^{-1}}^{(i)}, z]\}.\end{aligned}$$

Thus, if for each $h\in G$

$$\varepsilon_{g,h}\varepsilon_{gh,g^{-1}} \text{ is constant for all } g\in G \tag{5.32}$$

we have

$$[C^\varepsilon, z]=0 \tag{5.33}$$

and C^ε is a central element of $U^\varepsilon(L)$.

6. Contractions of $\mathbb{Z}_n$-graded representations of Lie algebras

Consider any $\mathbb{Z}_n$-graded Lie algebra L,

$$L=L_0\oplus\ldots\oplus L_{n-1} \tag{6.1}$$

with none of the commutators of grading subspaces identically equal zero,

$$0\neq[L_j, L_k]\subseteq L_{j+k} \qquad j, k, j+k \pmod n. \tag{6.2}$$

Treat $\mathbb{Z}_n$ as a set and let $\mathbb{Z}_n$ act on $\mathbb{Z}_n$ by $j\cdot k:=j+k \pmod n$. Suppose that

$$V=V_0\oplus\ldots\oplus V_{n-1} \tag{6.3}$$

is $\mathbb{Z}_n$-graded L-module in $\mathbb{Z}_n$-Mod(L). Let $\varepsilon\in C^2(\mathbb{Z}_n, K)$.

The grading is preserved during a contraction. Hence just as in (2.13), the contracted transformation $L\overset{\psi}{\cdot} V$ is described in terms of LV before the contraction and the matrix $\psi\in\mathbb{C}^{n\times n}$ of the contraction parameters for V. In general, one has

$$L\overset{\psi}{\cdot} V=\begin{pmatrix}\psi_{00}L_0 & \cdot & \cdot & \cdot & \psi_{1n-1}L_1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \cdot & \psi_{jk}L_j & \psi_{j-1,k+1}L_{j-1} & \cdot & \cdot\\ \cdot & \psi_{j+1,k}L_{j+1} & \psi_{j,k+1}L_j & \cdot & \cdot\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \psi_{n-1,0}L_{n-1} & \cdot & \cdot & \cdot & \psi_{0,n-1}L_0\end{pmatrix}\begin{pmatrix}V_0\\ V_1\\ \vdots\\ \cdot\\ \vdots\\ V_{n-1}\end{pmatrix} \tag{6.4}$$

where the subscripts of ψ are read modulo n. An example, $n=2$, of (6.4) is (2.13). For a fixed ε, the matrix $\psi=(\psi_{jk})$ is a solution of the system of quadratic equations

(2.16) or, equivalently, (5.6). A useful strategy for solving such a large number of equations is described next, assuming that a contraction ε is fixed.

Among the equations of (2.16) consider first the n equations

$$\varepsilon_{00}\psi_{0m}=\psi_{0m}^2 \qquad 0\leq m<n. \tag{6.5}$$

For any contraction we have either $\varepsilon_{00}=0$ or $\varepsilon_{00}=1$. In the first case $\psi_{0m}=0$ for $0\leq m<n$, in the second one $\psi_{0m}=0$ or 1.

With a fixed solution of equations (6.5) one considers the subset

$$\varepsilon_{0j}\psi_{jm}=\psi_{jm}\psi_{0,j+m}=\psi_{0m}\psi_{jm} \qquad 0<j<n, 0\leq m<n \tag{6.6}$$

of equations (2.16) from which some parameters ψ_{jm} are not determined at all and some are found to be zero. More precisely,

$$\psi_{j,m}=\begin{cases}\text{arbitrary} & \text{if } \varepsilon_{0j}=\psi_{0,j+m}=\psi_{0m}\\ 0 & \text{otherwise}\end{cases} \tag{6.7}$$

for $0<j<n$. Solutions of (6.5) and (6.7) are then used to simplify the remaining equations of the system (2.16) before solving them directly.

Let us now determine $\mathbb{Z}_3$-contractions of V. The $\mathbb{Z}_3$-contractions L^ε of L were found in [15]. There are eight continuous contractions

$$\varepsilon=\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}1&1&1\\1&0&0\\1&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$$
$$\begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \tag{6.8}$$

and five discrete ones

$$\varepsilon=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$$
$$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&1\\0&0&0\\1&0&0\end{pmatrix}. \tag{6.9}$$

In most cases of interest the grading subspaces L_1 and L_2 of $\mathbb{Z}_3$-grading of L could be interchanged, therefore we will consider only one of each pair of cases in (6.8) and (6.9) which differ by interchange $L_1\leftrightarrow L_2$.

The system of equations (2.16) which we have to solve for $n=3$ consists of 27 equations of the following form (for $m=0, 1, 2$):

$$\varepsilon_{00}\psi_{0m}=\psi_{0m}^2 \tag{6.10a}$$

$$\varepsilon_{01}\psi_{1m}=\psi_{1m}\psi_{0,m+1}=\psi_{1m}\psi_{0m} \tag{6.10b}$$

$$\varepsilon_{02}\psi_{2m}=\psi_{2m}\psi_{0,m+2}=\psi_{2m}\psi_{0m} \tag{6.10c}$$

$$\varepsilon_{11}\psi_{2m}=\psi_{1m}\psi_{1,m+1} \tag{6.10d}$$

$$\varepsilon_{22}\psi_{1m}=\psi_{2m}\psi_{2,m+2} \tag{6.10e}$$

$$\varepsilon_{12}\psi_{0m}=\psi_{2m}\psi_{1,m+2}=\psi_{1m}\psi_{2,m+1}. \tag{6.10f}$$

The system of equations (6.10) is invariant under the cyclic permutations of the columns of ψ. Its solutions are shown in table 2.

For $n=3$ we have from (6.4)

$$\begin{pmatrix} \psi_{00}L_0 & \psi_{21}L_2 & \psi_{12}L_1 \\ \psi_{10}L_1 & \psi_{01}L_0 & \psi_{22}L_2 \\ \psi_{20}L_2 & \psi_{11}L_1 & \psi_{02}L_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}=\begin{pmatrix} \psi_{00}L_0V_0+\psi_{21}L_2V_1+\psi_{12}L_1V_2 \\ \psi_{10}L_1V_0+\psi_{01}L_0V_1+\psi_{22}L_2V_2 \\ \psi_{20}L_2V_0+\psi_{11}L_1V_1+\psi_{02}L_0V_2 \end{pmatrix}. \tag{6.11}$$

The solutions of (6.10a) are listed in the following table with $m=0,1,2$.

Case	ε_{00}	$\psi_{0,m}$	$\psi_{0,m+1}$	$\psi_{0,m+2}$
a	0	0	0	0
b	1	1	1	1
c	1	1	0	0
d	1	0	1	1
e	1	0	0	0

(6.12)

The solutions of (6.10b) and (6.10c) are given in (6.7).

The matrices ε of (6.8) and (6.9), used in (6.10), together with (6.12) and (6.7) for $n=3$ lead to the following simplified versions of equations (6.10d)-(6.10f) with $m=0,1,2$:

$$\left.\begin{aligned} \psi_{2m}&=\psi_{1m}\psi_{1,m+1} \\ 0&=\psi_{2m}\psi_{2,m+2} \\ 0&=\psi_{2m}\psi_{1,m+2} \end{aligned}\right\}\qquad \text{solutions: } A,\ C,\ E,\ F \tag{6.13a}$$

$$\left.\begin{aligned} 0&=\psi_{1m}\psi_{1,m+1} \\ 0&=\psi_{2m}\psi_{2,m+2} \\ 0&=\psi_{2m}\psi_{1,m+2} \end{aligned}\right\}\qquad \text{solutions: } A,\ B,\ C,\ D,\ E \tag{6.13b}$$

$$\left.\begin{aligned} 0&=\psi_{1m} \\ 0&=\psi_{2m}\psi_{2,m+2} \\ 0&=\psi_{2m}\psi_{1,m+2} \end{aligned}\right\}\qquad \text{solutions: } A,\ B \tag{6.13c}$$

$$\{1\leftrightarrow 2 \text{ in } (6.13c)\}\qquad \text{solutions: } A,\ C. \tag{6.13d}$$

The solutions $A, B, \ldots, E$ in (6.13) are the following matrices $\psi=(\psi_{jk})$:

$$A=\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\qquad B=\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}\qquad C=\begin{pmatrix} * & * & * \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} * & * & * \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}\qquad E=\begin{pmatrix} * & * & * \\ \alpha & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}\qquad F=\begin{pmatrix} * & * & * \\ \alpha & \beta & 0 \\ \alpha\beta & 0 & 0 \end{pmatrix}. \tag{6.14}$$

The top row is arbitrary because it does not enter (6.13). Also α and β are different from 0 but otherwise are any complex numbers. The matrices B, C, D, E, F with *cyclically permuted columns* are also solutions of (6.13).

Table 2. The $\mathbb{Z}_3$-graded contractions of representations of $\mathbb{Z}_3$-graded contractions of Lie algebras. The Lie algebra contractions is given by ε. The contraction of representation is given by the corresponding ψ or, equivalently, by $L \overset{\psi}{\cdot} V$ in the format of equation (6.11). Out of three matrices ψ which differ by cyclic permutation of columns only one is shown.

ε	No.	ψ	$L \overset{\psi}{\cdot} V$
$\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}$	I.1	$\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\L_2&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.2	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&L_2\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.3	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.4	$\begin{pmatrix}1&1&1\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\L_2&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.5	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.6	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.7	$\begin{pmatrix}1&1&0\\0&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}L_0&L_2&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.8	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	I.9	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}$	II.1	$\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\L_2&L_1&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	II.2	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	II.3	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	II.4	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	II.5	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$

Table 2. (continued)

ε	No.	ψ	$L \overset{\psi}{\cdot} V$
	II.6	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	III.1	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&L_2&L_1\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	III.2	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	IV.1	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&L_2&L_1\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	IV.2	$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	V.1	$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&L_2&L_1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	V.2	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&0\\L_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	V.3	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	V.4	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\L_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	VI.1	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.2	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.3	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.4	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&L_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.5	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$

Table 2. (continued)

ε	No.	ψ	$L \overset{\psi}{\cdot} V$
	VI.6	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&0\\L_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.7	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&L_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.8	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VI.9	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\L_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	VII.1	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VII.2	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VII.3	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VII.4	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VII.5	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&L_2&L_1\\0&0&0\\0&L_1&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VII.6	$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\L_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	VIII.1	$\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VIII.2	$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&L_0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VIII.3	$\begin{pmatrix}1&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\L_1&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VIII.4	$\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&L_0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$

Table 2. (continued)

ε	No.	ψ	$L^{\psi}\cdot V$
	VIII.5	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VIII.6	$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}L_0&0&0\\0&0&L_2\\0&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$
	VIII.7	$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\L_2&0&0\end{pmatrix}\begin{pmatrix}V_0\\V_1\\V_2\end{pmatrix}$

There remains the case-by-case analysis of the individual contractions and further restriction of the range of values of α and β in (6.14) by renormalization of subspaces V_m, $m=0, 1, 2$, according to (5.20) and possibly also the grading subspaces L_k, $k=0$, 1, 2, according to (3.11). We illustrate this next in one case; table 2 contains all the results.

Example 2. Consider the contraction L^{ε} given by

$$\varepsilon=\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}.$$

In this case (6.10) becomes

$$\begin{aligned}
\psi_{0,m}&=\psi_{0m}^2\\
\psi_{1,m}&=\psi_{1m}\psi_{0,m+1}=\psi_{1m}\psi_{0m}\\
\psi_{2,m}&=\psi_{2m}\psi_{0,m+2}=\psi_{2m}\psi_{0m}\\
0&=\psi_{1m}\psi_{1,m+1}\\
0&=\psi_{2m}\psi_{2,m+2}\\
0&=\psi_{2m}\psi_{1,m+2}.
\end{aligned}\tag{6.15}$$

First put $\psi_{0m}=1$ for all m, which is case b of (6.12). This reduces (6.15) to the last three equations, i.e. to the system (6.13b) whose solutions are A, B, C, D, E of (6.14) together with the cyclic permutation of columns in B, C, D, E. Let us write out one solution of type E corresponding to $m=1$. In that case we have ψ as the matrix E with one step cyclic permutation of columns:

$$\psi=\begin{pmatrix}1&1&1\\0&\alpha&0\\\beta&0&0\end{pmatrix}.\tag{6.16}$$

In the form (6.11) we write the contracted graded linear transformations as

$$\begin{pmatrix} L_0 & 0 & 0 \\ 0 & L_0 & 0 \\ \beta L_2 & \alpha L_1 & L_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}=\begin{pmatrix} L_0 V_0 \\ L_0 V_1 \\ \beta L_2 V_0+\alpha L_1 V_1+L_0 V_2 \end{pmatrix}. \tag{6.17}$$

Let us compare (6.17) with similar solutions for $m=0$ and 2 respectively:

$$\begin{pmatrix} L_0 & 0 & 0 \\ \alpha L_1 & L_0 & \beta L_2 \\ 0 & 0 & L_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}=\begin{pmatrix} L_0 V_0 \\ \alpha L_1 V_0+L_0 V_1+\beta L_2 V_2 \\ L_0 V_2 \end{pmatrix} \tag{6.18}$$

$$\begin{pmatrix} L_0 & \beta L_2 & \alpha L_1 \\ 0 & L_0 & 0 \\ 0 & 0 & L_0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}=\begin{pmatrix} L_0 V_0+\beta L_2 V_1+\alpha L_1 V_2 \\ L_0 V_1 \\ L_0 V_2 \end{pmatrix}. \tag{6.19}$$

One can verify directly that (6.17)-(6.19) are, indeed, representations of the contracted Lie algebras L^ε. Let us do this just for the commutators

$$[L_0, L_1]_\varepsilon \subseteq L_1 \qquad [L_1, L_2]_\varepsilon = 0$$

and for the representation (6.17). We have therefore L_0, L_1 and L_2 respectively as the matrices

$$\begin{pmatrix} L_0 & 0 & 0 \\ 0 & L_0 & 0 \\ 0 & 0 & L_0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta L_2 & 0 & 0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned}
[L_0, L_1]_\varepsilon \overset{\psi}{\cdot} V &= (L_0 L_1 - L_1 L_0)_\varepsilon \overset{\psi}{\cdot} V \\
&= \left\{\begin{pmatrix} L_0 & 0 & 0 \\ 0 & L_0 & 0 \\ 0 & 0 & L_0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 & 0 \end{pmatrix}\begin{pmatrix} L_0 & 0 & 0 \\ 0 & L_0 & 0 \\ 0 & 0 & L_0 \end{pmatrix}\right\}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_0 L_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 L_0 & 0 \end{pmatrix}\right\}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha [L_0, L_1] & 0 \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \\
&= \begin{pmatrix} V_0 \\ \alpha [L_0, L_1] V_1 \\ V_2 \end{pmatrix}.
\end{aligned}$$

Similarly, we find

$$[L_1, L_2]V = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta L_2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta L_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha L_1 & 0 \end{pmatrix}\right\}\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The parameters α and β in (6.17), (6.18) and (6.19) can be transformed to 1 without loss of generality, if they are different from zero. That is achieved by renormalizing the corresponding grading subspaces.

Among the representations (6.17)-(6.19), only one of them with $\alpha = \beta = 1$ is shown in table 2. In order to complete the contraction of representations of L^ε in our example, one would first need to complete the case b of (6.12) by considering also the solutions of types A, B, C, D in addition to E, and then find the solutions for the cases c, d, e of (6.12).

Example 3. Consider an example of a $\mathbb{Z}_3$-graded affine Lie algebra $\hat{A}_1$ generated by

$$e_{(n)} = e \otimes t^n \qquad f_{(n)} = f \otimes t^n \qquad h_{(n)} = h \otimes t^n \qquad \not{c} \tag{6.20}$$

satisfying the commutation relations

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m\delta_{m+n,0}(x|y)\not{c} \tag{6.21}$$

where $(x|y)$ is the Killing form in the three-dimensional simple Lie algebra generated by e, f, h with the commutation rules

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f. \tag{6.22}$$

We choose the $\mathbb{Z}_3$-grading subspaces L_0, L_1 and L_2 of $\hat{A}_1$ spanned by the following generators,

$$\begin{aligned} L_0 &= \{h_{(3k)}, e_{(3k+2)}, f_{(3k+1)}, \not{c}\} \\ L_1 &= \{h_{(3k+1)}, e_{(3k)}, f_{(3k+2)}\} \\ L_2 &= \{h_{(3k+2)}, e_{(3k+1)}, f_{(3k)}\} \qquad -\infty < k < \infty. \end{aligned} \tag{6.23}$$

We make the usual identification with the generators E_0, E_1, H_0, H_1, F_0, F_1 of the affine algebra

$$E_1 = e_{(0)} \in L_1 \qquad H_1 = h_{(0)} \in L_0 \qquad F_1 = f_{(0)} \in L_2$$

$$E_0 = f_{(1)} \in L_0 \qquad H_0 = \not{c} - h_{(0)} \in L_0 \qquad F_0 = e_{(-1)} \in L_1.$$

Let us take two irreducible representations of $\hat{A}_1$ with highest weights (1 0) and (1 1). The corresponding representation space V decomposes under the action of L_0, L_1 and L_2 into subspaces V_0, V_1 and V_2 respectively. Our task is to describe V_0, V_1 and V_2 for the two representations (1 0) and (1 1) and then use them in some of the cases listed in table 2. We choose the example considered in (6.17).

The space V is an infinite direct sum of finite-dimensional subspaces labelled by weights of the corresponding representation. A practical way to give V_0, V_1 and V_2 is to indicate the weight subspaces spanning them.

The first few weights of (1 0) and (1 1) are shown in figure 1 in the standard basis of fundamental weights.

The horizontal lines in figure 1 linking two boxes indicate transformations by $f_{(0)}$ (from left to right) and $e_{(0)}$ (from right to left). The vertical connecting lines are due to $e_{(-1)}$ (direction down) and $f_{(1)}$ (direction up). Since both directions are valid transformations, the lines on figure 1 are not oriented. Therefore one can immediately

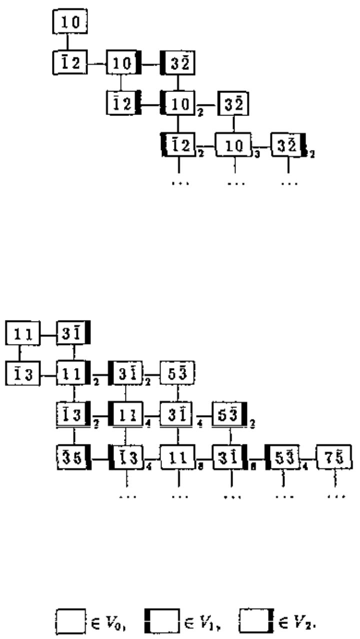

Figure 1. The first few weights of the affine-A_1 representations with highest weights (1 0) and (1 1). The subscripts indicate the multiplicity of the weight when it is greater than 1; the overbar is the minus sign.

indicate which weights of figure 1 label subspaces of V_0, V_1 and V_2. However, one of them, the highest weight say, has to be assigned arbitrarily (let it be in V_0).

The contraction of our example is that of (6.17). Hence, we have the annihilating actions

$$L_1 \overset{\psi}{\cdot} V_2 = L_1 \overset{\psi}{\cdot} V_0 = L_2 \overset{\psi}{\cdot} V_1 = L_2 \overset{\psi}{\cdot} V_2 = 0.$$

Now we can redraw figure 1 with the lines correspondingly erased. If transformation by $e_{(0)}$, $f_{(0)}$, $e_{(-1)}$, $f_{(1)}$ is non-zero in one direction only, the corresponding line is oriented. Results for (1 0) and (1 1) are shown in figure 2. Let us emphasize that there are many other transformations possible due to generators other than the four simple ones between weight subspaces, but these are not shown in either figure 1 or 2.

After the contraction both representations remain indecomposable. Figure 2 does not quite reveal this fact. Note that with the contracted commutation, E_0 and E_1 do not generate the positive part of L. Indeed, $[E_1, [E_1, E_0]_\varepsilon]_\varepsilon = 0$.

7. Contractions of tensor product decompositions

Let ε define a contraction for G-Lie and let V and W be L-modules for some $L \in G$-Lie. Suppose that V and W are graded by M and M is a *group* (we use multiplicative notation). We suppose that the G-action on M satisfies

$$g \cdot (m \cdot n) = (g \cdot m) \cdot n \qquad \text{for all } g \in G,\ m, n \in M. \tag{7.1}$$

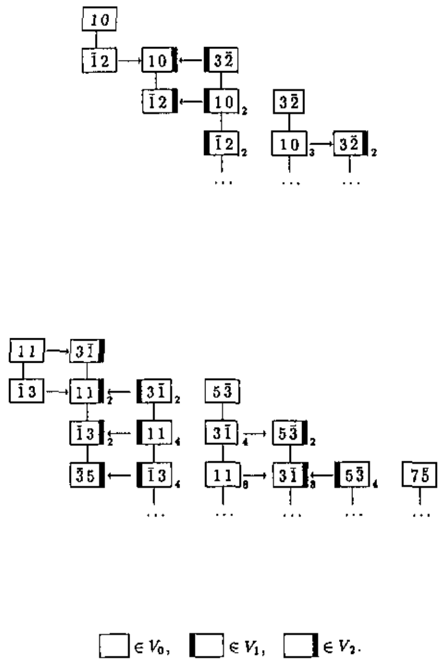

Figure 2. The oriented horizontal and vertical lines indicate the non-zero action of $e_{(0)}$, $f_{(0)}$, $e_{(-1)}$, $f_{(1)}$, on the weight subspaces. The $\mathbb{Z}_3$-grading of figure 1 is preserved here; the overbar is the minus sign.

Let ψ be a contraction of V compatible with ε. We denote V considered as an L^{ε}-module by V^{ψ}. Consider $V\otimes W$ graded by M in the usual way:

$$(V\otimes W)_p := \sum_{mn=p} V_m \otimes W_n.$$

Thus we may also contract $V\otimes W$ to get $(V\otimes W)^{\psi}$. We wish to compare $(V\otimes W)^{\psi}$ with all $V^{\psi}\otimes W^{\psi}$. Both are L^{ε}-modules. Let $v\in V_m$, $w\in W_n$, $x\in L_g$. Then $v\otimes w\in (V\otimes W)^{\psi}_{mn}$ and

$$x\cdot(v\otimes w)=\psi_{g,mn}x(v\otimes w)=\psi_{g,mn}((xv\otimes w)+(v\otimes xw)). \tag{7.2}$$

However $v\otimes w\in V^{\psi}_m\otimes W^{\psi}_n$. If we use the tensor product action of L^{ε} on $v\otimes w$ seen in this way we get

$$x\cdot(v\otimes w)=x\cdot v\otimes w+v\otimes x\cdot w=\psi_{g,m}xv\otimes w+\psi_{g,n}v\otimes xw. \tag{7.3}$$

Unless $\psi_{g,mn}=\psi_{g,m}=\psi_{g,n}$, (7.2) and (7.3) will not in general be equal.

In order to compare $(V\otimes W)^{\psi}$ with $V^{\psi}\otimes W^{\psi}$ we need an L^{ε}-map between them. The above tells us that straight identification will not work without extremely restrictive hypotheses on ψ. Instead we try to construct a map

$$\tau: V^{\psi}\otimes W^{\psi}\to V\overset{\tau}{\otimes} W \tag{7.4}$$

$$\tau|_{(V^{\psi}_m\otimes W^{\psi}_n)}=\tau_{m,n} \qquad \tau_{m,n}\in K. \tag{7.5}$$

We want τ to be a L^{ε}-map. Let v, w, x be as above, then

$$\begin{aligned} x\cdot\tau(v\otimes w)&=x\cdot v\overset{\tau}{\otimes}w=x\cdot\{\tau_{m,n}v\otimes w\}\\ &=\tau_{m,n}\psi_{g,mn}x(v\otimes w)\\ &=\tau_{m,n}\psi_{g,mn}(xv\otimes w+v\otimes xw) \end{aligned}\tag{7.6}$$

$$\begin{aligned} \tau(x\cdot(v\otimes w))&=\tau(x\cdot v\otimes w+v\otimes x\cdot w)\\ &=\psi_{g,m}(xv\overset{\tau}{\otimes}w+\psi_{g,n}v\otimes xw)\\ &=\psi_{g,m}\tau_{gm,n}xv\otimes w+\psi_{g,n}\tau_{m,gn}v\overset{\tau}{\otimes}xw. \end{aligned}\tag{7.7}$$

Comparing (7.6) and (7.7) we must have (generically)

$$\tau_{m,n}\psi_{g,mn}=\psi_{g,m}\tau_{gm,n}=\psi_{g,n}\tau_{m,gn}.\tag{7.8}$$

In particular, equations (7.8) impose no restriction on τ if the only non-zero matrix elements of ψ are those given by $\psi_{1,h}=1$, for any $h\in M$. In all other cases τ is symmetric, $\tau=\tau^{\mathrm{T}}$.

An interesting example of this is the case $M=G$, $\psi=\varepsilon=\tau$.

In the case of $\mathbb{Z}_2$-grading the solutions τ of (7.8) are summarized in table 3.

Consider a decomposition $V\otimes W=\bigoplus U(i)$ as L-modules. Assume that each $U(i)$ is M-graded by inheritance from $V\otimes W$, i.e.

$$U(i)=\bigoplus_{m\in M}\{U(i)\cap(V\otimes W)_m\}.\tag{7.9}$$

Table 3. Non-trivial $\mathbb{Z}_2$-graded contractions of tensor products of two representations are given in terms of the matrices τ, solutions of (7.8), for fixed ε (contraction of the Lie algebra) and fixed ψ (contraction of representations).

ψ	τ	ε
$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$
$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	Arbitrary	$\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}$
$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}$
$\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}$
$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}$
$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}$

Then $U(i)^\psi$ exists. Note that for $u\in U(i)^\psi_m\subset (V\otimes W)^\psi$, and for $x\in L^\varepsilon_g$,

$$x\cdot u=\psi_{g,m}xu\in U(i)^\psi_{gm}.$$

Thus each $U(i)^\psi$ is an L^ε-module. Therefore

$$(V\otimes W)^\psi=\bigoplus U(i)^\psi \tag{7.10}$$

as L^ε-modules.

Now if we denote the image of τ by I then $I\cap U(i)^\psi$ is an L^ε-submodule of $(V\otimes W)^\psi$ and $P(i):=\tau^{-1}(I\cap U(i)^\psi)$ is an L^ε-submodule of $V^\psi\otimes W^\psi$. Set

$$P:=\sum P(i) \qquad \text{and} \qquad C:=\tau^{-1}(0)=\sum V_m\otimes W_n \qquad \tau_{m,n}=0. \tag{7.11}$$

Then we have the sequence of L^ε-submodules

$$V^\psi\otimes W^\psi\supset P\supset C\supset(0) \tag{7.12}$$

and

$$P/C\simeq\bigoplus\{(P(i)+C)/C\}. \tag{7.13}$$

Example 4. We consider what this looks like in the case of $\mathbb{Z}_2$-gradings. Suppose that V and W are $\mathbb{Z}_2$-graded L-modules. Adopting the matrix notation as in section 2, we introduce the grading structure explicitly also for the tensor product

$$V\otimes W=:\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}=\begin{pmatrix}V_0\otimes W_0+V_1\otimes W_1 & V_1\otimes W_0+V_0\otimes W_1\\ V_1\otimes W_0+V_0\otimes W_1 & V_0\otimes W_0+V_1\otimes W_1\end{pmatrix}$$

where, as in section 2, we could collapse the last matrix to

$$\begin{pmatrix}V_0\otimes W_0+V_1\otimes W_1\\ V_0\otimes W_1+V_1\otimes W_0\end{pmatrix}$$

if no further operations are required. We express the action of τ on $V\otimes W$ by

$$V\otimes W\to V\overset{\tau}{\otimes} W=\begin{pmatrix}\tau_{00}V_0\otimes W_0+\tau_{11}V_1\otimes W_1 & \tau_{01}V_0\otimes W_1+\tau_{10}V_1\otimes W_0\\ \tau_{01}V_0\otimes W_1+\tau_{10}V_1\otimes W_0 & \tau_{00}V_0\otimes W_0+\tau_{11}V_1\otimes W_1\end{pmatrix}.$$

Then we have

$$L\overset{\psi}{\cdot}((V\otimes W)^\tau)=(L\overset{\psi}{\cdot}(V\otimes W))^\tau$$

given in terms of matrix multiplications

$$\begin{aligned}L\overset{\psi}{\cdot}&\left(\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\overset{\tau}{\otimes}\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}\right)\\ &=\left\{\left(L\overset{\psi}{\cdot}\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\right)\otimes\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}+\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\otimes L\overset{\psi}{\cdot}\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}\right\}^\tau\\ &=L\overset{\psi}{\cdot}\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\overset{\tau}{\otimes}\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}+\begin{pmatrix}V_0 & V_1\\ V_1 & V_0\end{pmatrix}\overset{\tau}{\otimes}L\overset{\psi}{\cdot}\begin{pmatrix}W_0 & W_1\\ W_1 & W_0\end{pmatrix}.\end{aligned}$$

Consider the contraction defined by $\varepsilon=\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$ and $\tau=\psi=\varepsilon$. Then the matrix notation in parallel,

$$I=V\overset{\tau}{\otimes}W=V_0\otimes W_0+V_0\otimes W_1+V_1\otimes W_0=\begin{pmatrix}V_0\otimes W_0\\ V_0\otimes W_1+V_1\otimes W_0\end{pmatrix} \tag{7.14}$$

Suppose that we have a decomposition $V\otimes W=\bigoplus U(i)$, where each $U(i)$ is $\mathbb{Z}_2$-graded from $V\otimes W$,

$$U(i)=\begin{pmatrix}U(i)_0\\ U(i)_1\end{pmatrix}.$$

Then $U(i)_1\subset I$ and

$$\begin{aligned}P(i)\supset U(i)_1+V_1\otimes W_1&=\begin{pmatrix}0\\ U(i)_1\end{pmatrix}+\begin{pmatrix}V_1\otimes W_1\\ 0\end{pmatrix}\\ &=\begin{pmatrix}V_1\otimes W_1\\ U(i)_1\end{pmatrix}=:P(i)'.\end{aligned} \tag{7.15}$$

In general, $P(i)'\neq P(i)$. However $P(i)'$ is a L^{ε}-module since

$$U(i)_1\subset V_0\otimes W_1+V_1\otimes W_0 \qquad L_1^{\varepsilon}(V_0\otimes W_1+V_1\otimes W_0)\subset V_1\otimes W_1.$$

Set

$$\begin{aligned}P'&:=\sum P(i)'=\sum U(i)_1+V_1\otimes W_1\\ &=(V\otimes W)_1+V_1\otimes W_1\\ &=V_0\otimes W_1+V_1\otimes W_0+V_1\otimes W_1\\ &=\begin{pmatrix}V_1\otimes W_1\\ V_1\otimes W_0+V_1\otimes W_1\end{pmatrix}.\end{aligned} \tag{7.16}$$

Thus we have the chain of submodules

$$\begin{array}{c} V^{\psi}\otimes W^{\psi}\\ |\\ P'=\sum P(i)'\\ |\\ C\\ |\\ 0\end{array}\quad \begin{array}{l}\}\simeq V_0\otimes W_0\\ \}\simeq \oplus\overline{U(i)}\\ \}\simeq V_1\otimes W_1\end{array} \tag{7.17}$$

where $\overline{U(i)}:=(U(i)+C)/C$. The three indicated quotient modules are all L^{ε}-modules on which L_1^{ε} acts trivially. Thus we obtain considerable information about the tensor product of the contracted representations.

Let us look at a particular example of this, namely $L=\mathrm{sl}(4,\mathbb{C})$ with the $\mathbb{Z}_2$-grading defined by assigning degree 0 to the simple roots $\pm\alpha_1$ and $\pm\alpha_2$ and degree 1 to the simple roots $\pm\alpha_3$. Then $L_0=\mathrm{sl}(3,\mathbb{C})\times u$ where u is one-dimensional and L^{ε} has a Levi decomposition

$$\mathrm{sl}(3,\mathbb{C})\oplus\Big(u\oplus\sum_{\alpha\in\Delta_1}L^{\alpha}\Big) \tag{7.18}$$

where

$$\Delta_1:=\Big\{\alpha\in\Delta\,\Big|\,\alpha=\sum c_i\alpha_i,\ c_3\equiv 1 \bmod 2\Big\}.$$

Any irreducible representation in congruence class 0 (weight system inside the root lattice of $\mathrm{sl}(4,\mathbb{C})$) is compatibly $\mathbb{Z}_2$-graded and has an ε-contraction. For example the

Table 4. Branching of $sl(4,\mathbb{C})$ representations to $sl(3,\mathbb{C})\times u$ representations. The entries in the left column are $sl(4,\mathbb{C})$ weight labels and those in the top row are $sl(3,\mathbb{C})$ weight labels. The table entries consist of the $sl(3,\mathbb{C})$ multiplicities together with the appropriate u-labels. Thus for instance in (2 1 0) the representation (2 0; −4) occurs once.

	Even					Odd					
	(2 2)	(0 3)	(3 0)	(1 1)	(0 0)	(1 2)	(2 1)	(0 2)	(2 0)	(0 1)	(1 0)
(2 0 2)	1; 0			1; 0	1; 0	1; −4	1; +4	1; −8	1; +8	1; −4	1; +4
(0 1 2)		1; 0		1; 0		1; −4		1; +4		1; +8	1; +4
(2 1 0)			1; 0	1; 0			1; +4		1; −4	1; −4	1; −8
2(1 0 1)				2; 0	2; 0					2; −4	2; +4
(0 2 0)				1; 0				1; +4	1; −4		
(0 0 0)					1; 0						

15-dimensional representation with weight labels (1 0 1) decomposes relative to L_0 as

$$\underbrace{(1,1;0)+(0,0;0)}_{\substack{(1\,0\,1)_0\\ \dim 9}}+\underbrace{(1,0;4)+(0,1;-4)}_{\substack{(1\,0\,1)_1\\ \dim 6}}. \tag{7.19}$$

In order to determine (7.17) we decompose $(1\,0\,1)\otimes(1\,0\,1)$ as an $sl(4,\mathbb{C})$-module and decompose it as an L_0-module into even and odd parts. Thus

$$(1\,0\,1)\otimes(1\,0\,1)=(2\,0\,2)\oplus(0\,1\,2)\oplus(2\,1\,0)\oplus2(1\,0\,1)\oplus(0\,2\,0)\oplus(0\,0\,0)$$

decomposes according to table 4, and

$$P'=\left(\overbrace{(2,1;4)+(1,2;-4)+(2,0;-4)+(0,2;4)+2(1,0;4)+2(0,1;-4)}^{V_1\otimes W_1}\right)$$

in this example.

8. Concluding remarks

It is worthwhile pointing out again that the product of two contractions is also a contraction. For more complicated grading groups this provides a useful tool for producing new contractions from old ones.

In this paper we have assumed in developing our equations that we are in the generic case. This does not mean that the Lie algebras and representations involved must themselves be generic for the contractions to work. Rather the generic case imposes the maximum number of conditions on the parameters. Lie algebras and/or representations which are not generic will admit contractions beyond those appearing in our classification.

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